Matrix formulation for the propagation of light beams with orbital and spin angular momenta

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Jones matrices describe the polarization, or spin angular momentum, of a light beam as it passes through an optical system. We devise an equivalent of the Jones matrix formulation for light possessing orbital angular momentum. The matrices are then developed to account for light that has both spin and orbital angular momentum. [S1063-651X(99)07412-7]

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The behavior of polarized light as it passes through birefringent optical components is well described by an approach devised by Jones [1]. The beam is represented in terms of a column vector, the Jones vector, the elements of which are the complex amplitudes of the orthogonally polarized components. Optical components such as linear and circular polarizers, phase retarders, etc., are represented by 2×2 matrices called Jones matrices: matrix multiplication accounts for the passage of the light through a sequence of optical elements. An alternative formulation derived by Mueller [2], based on the Stokes parameters, is not able to deal fully with the superposition of coherent beams, but has the advantage of being applicable to partially as well as fully polarized light. A full account of both the Jones and Mueller matrices and their application, has been given by Gerrard and Burch [3].

Polarized light is associated with spin angular momentum. It is now well established [4–10] that light beams may possess well defined orbital angular momentum. Beams with an azimuthal phase dependence of $\exp(il\phi)$, such as Laguerre Gaussian beams, have an orbital angular momentum of $l\hbar$ per photon [4]. Such modes may be readily created in the laboratory by passing the Hermite Gaussian modes usually emitted by lasers, through a mode converter [5] which consists of two canonically disposed cylindrical lenses oriented at 45° to the axes of the mode. The indices (n, m) characterizing the Hermite Gaussian modes give the indices (l, p) of the Laguerre Gaussian modes, where $l=|n-m|$ and $p=\min(m,n)$ [4].

Hermite Gaussian and Laguerre Gaussian modes both form complete, orthogonal, basis sets from which any arbitrary field distribution may be described. The order of a mode is defined [5] by $N=n+m=2p+|l|$ and does not change as the mode is converted or rotated. There is an exact analogy between a waveplate for polarized light and a mode converter; see Fig. 1. Just as a phase shift is introduced between orthogonal polarization components by birefringent waveplates, so mode converters based on cylindrical lenses introduce a phase shift between orthogonal modes of the same order. This equivalence allows polarization and mode structure to be treated in similar ways and leads to a formulation, analogous to the Jones matrix approach, for modes of order N.

The Poincaré sphere is an equivalent representation to the Jones matrix formulation for polarized light. Any state of polarization may be represented by a point on the sphere; waveplates and other polarizing elements then move the polarization state to another position on the surface of the sphere. It has been shown [11], that there exists an orbital angular momentum equivalent to the Poincaré sphere for modes of order $N=1$. A Poincaré sphere approach to arbi-

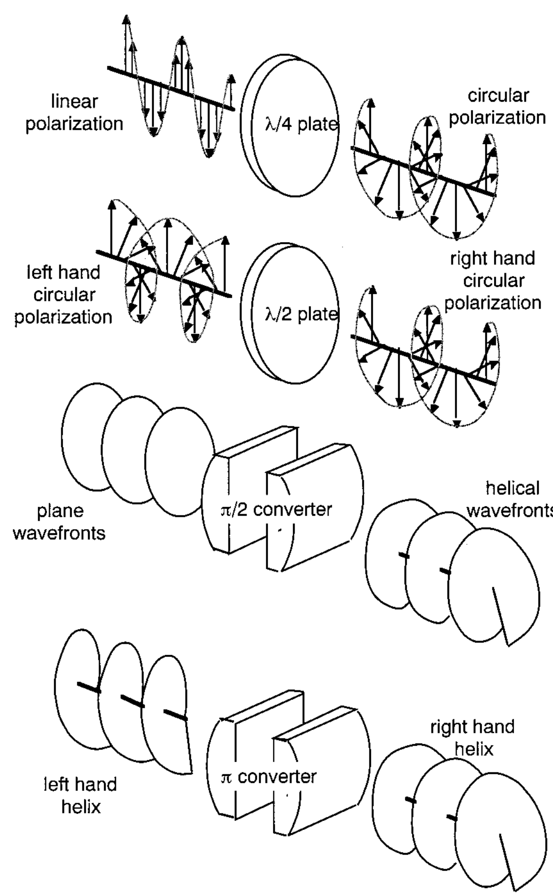

FIG. 1. Quarter and half wave plates and $\pi/2$ or π mode converters play equivalent roles for spin and orbital angular momentum.

TABLE I. $[(N+1)\times(N+1)]$ matrices for modes of order N.

$\pi/2$ converter

$$[C(\pi/2)]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & -i & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & -1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & i & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & -i & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

π converter

$$[C(\pi)]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & -1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & -1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Mode filter (e.g. $HG_{N-2,2}$)

$$[F(N-2,2)]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Mode rotation matrix for $N=1$

$$[\mathrm{rot}(\phi)]=\begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$$

Mode rotation matrix for $N=2$

$$[\mathrm{rot}(\phi)]=\begin{bmatrix} \cos^2\phi & \frac{\sin 2\phi}{\sqrt{2}} & \sin^2\phi \\ \frac{-\sin 2\phi}{\sqrt{2}} & \cos 2\phi & \frac{\sin 2\phi}{\sqrt{2}} \\ \sin^2\phi & \frac{-\sin 2\phi}{\sqrt{2}} & \cos^2\phi \end{bmatrix}$$

trary order of N is, however, not easy to devise as higher order geometries are required. This is not a difficulty for a matrix formulation.

In this paper we show that just as there is a two element Jones column vector and a set of 2×2 matrices for polarized light, so there is an $(N+1)$-element column vector and a set of $[(N+1)\times(N+1)]$ matrices to describe the passage of modes of order N, which may possess orbital angular momentum, through a series of optical components such as mode converters and beam rotators. For these matrices to apply, N must be conserved and the modes must have the same Rayleigh range and beam waist. Most imaging systems

TABLE I (*continued*).

Mode rotation matrix for $N=3$
$$[\mathrm{rot}(\phi)]=\begin{bmatrix} \cos^3\phi & \sqrt{3}\cos^2\phi\sin\phi & \sqrt{3}\cos\phi\sin^2\phi & \sin^3\phi \\ -\sqrt{3}\cos^2\phi\sin\phi & \dfrac{\cos\phi+3\cos 3\phi}{4} & \dfrac{-\sin\phi+3\sin 3\phi}{4} & \sqrt{3}\cos\phi\sin^2\phi \\ \sqrt{3}\cos\phi\sin^2\phi & \dfrac{\sin\phi-3\sin 3\phi}{4} & \dfrac{\cos\phi+3\cos 3\phi}{4} & \sqrt{3}\cos^2\phi\sin\phi \\ -\sin^3\phi & \sqrt{3}\cos\phi\sin^2\phi & -\sqrt{3}\cos^2\phi\sin\phi & \cos^3\phi \end{bmatrix}$$

satisfy this requirement, but those with optical components which may result in energy exchange between modes such as optical fibres, holograms apertures and nonlinear crystals [9], will not.

Any mode of order N can be expanded as the sum of $(N+1)$ Hermite Gaussian, or Laguerre Gaussian, modes of the same order [4,5]. It follows that such a mode may be represented by a column vector with $(N+1)$ elements

$$\begin{bmatrix} a_{N,0} \\ a_{N-1,0} \\ \cdots \\ \cdots \\ a_{1,N-1} \\ a_{0,N} \end{bmatrix},$$

where $a_{n,m}$ is the complex amplitude coefficient of the (n, m) Hermite Gaussian mode. For example, a Hermite Gaussian (2, 0) mode oriented at 45° to the laboratory frame is represented by a three element column vector containing the complex amplitude weightings of the (2,0), (1,1), (0,2) Hermite Gaussian modes, respectively. This rotated Hermite Gaussian mode is transformed to a Laguerre Gaussian, by passing the beam through a $\pi/2$ converter, which introduces phase shifts between the constituent modes of $(m-n)\pi/4$.

The matrices were developed by consideration of the role of each optical component. The way in which a $\pi/2$ converter transforms a (n,m) Hermite Gaussian mode to a (p,l) Laguerre Gaussian mode is well understood [5]. This converts a mode with no orbital angular momentum into one possessing it. The relative phases of the initial and final component modes are known and the nature of the required matrix follows easily enough. Likewise the matrix for a π converter, which converts a mode with orbital angular momentum l to one with $-l$, follows readily. It is rather more difficult to determine the form of the matrix for the rotation of a mode. Each element of the matrix can be deduced by considering the form of the decomposition of each of the individual modes as it is rotated through an angle ϕ. We may note that components of the higher orders of the rotation matrix follow from the transformation of angular momentum eigenfunctions under finite rotation, given by expressions (4.1.15) or (4.1.23), in the book by Edmonds [12] on angular momentum.

The optical component analogous to a polarizer is a mode filter which selects one constituent mode from those in the column vector. Such filters have been developed and employed to preferentially transmit specific modes [13]. The matrices for $\pi/2$ converters, π converters, mode filters and mode rotations are given in Table I.

To show how the matrices may be used, consider again the (2,0) Hermite Gaussian mode. The column vector representing such an input mode is that for mode order $N=n+m=2$. The vector, therefore, has three components. The first term in the column vector indicates that the input mode is indeed the (2,0) mode and the subsequent zeros show that there is no admixture of (1,1) or (0,2) modes. When the mode is oriented at 45° and passed through a $\pi/2$ converter, the column vector describing the output beam is given by

$$[C(\pi/2)]\times[\mathrm{rot}(45°)]\times[HG_{2,0}]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{bmatrix}\times\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}\times\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ \frac{-1}{2} \end{bmatrix}.$$

From Beijersbergen *et al.* [5], we recognize the Hermite Gaussian coefficients of this output beam as a Laguerre Gaussian mode with indices $l=2$ and $p=0$.

The coefficients of the column vectors in the Jones matrix formulation correspond, respectively, to vertically and horizontally polarized light. It would be possible to formulate an equivalent representation in which the coefficients were for right and left handed circular polarizations. In the same way, our beam description in terms of Hermite Gaussian modes could be replaced with a description in terms of Laguerre Gaussian modes. Conceptually there is no difference between the two approaches, except that in our formulation the matrices for optical components are simple and the rotation matrix relatively complicated while for a column vector based on Laguerre Gaussian modes, the inverse would be the case.

TABLE II. $[2(N+1)\times 2(N+1)]$ matrices for polarized modes of order N.

$\pi/2$ converter

$$[C(\pi/2)]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

π converter

$$[C(\pi)]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$\lambda/4$ waveplate

$$[W(\lambda/4)]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Light with a particular mode structure, and so a defined orbital angular momentum may also be polarized and have a defined spin angular momentum. In many instances the two attributes of the beam will remain totally uncoupled and the resultant beam may be determined by two independent calculations: one with the Jones matrices and the other with the matrices developed for orbital angular momentum. However, this is not always the case; for example in the calculation of the eigenmodes of beams within laser resonators ''out-of-plane'' ring geometries and birefringent optical components

TABLE II (*continued*).

λ/2 waveplate

$$[W(\lambda/2)]=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}$$

Mode filter (e.g., $HG_{N-2,2}$)

$$[F(N-2,2)]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}$$

Polarizer; for vertical polarization

$$[P(\updownarrow)]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}$$

give rise to rotations of the mode profile and state of polarization, respectively [14]. The corresponding eigenvalues relate to the round trip loss and it is unclear how they could be determined from independent calculation of the orbital and spin eigenmodes. Another example, which we analyze below, is the recently studied rotational frequency shift [10] where the orbital and spin angular momentum interact in such a way that the total angular momentum is the important parameter, not the individual contributions.

More generally, the study of the orbital angular momen-

TABLE II (*continued*).

Mode rotation matrix for $N=1$

$$[\mathrm{rot}_{\mathrm{mode}}(\phi)]=\begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) & 0 \\ 0 & \cos(\phi) & 0 & \sin(\phi) \\ -\sin(\phi) & 0 & \cos(\phi) & 0 \\ 0 & -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

Mode rotation matrix for $N=2$

$$[\mathrm{rot}_{\mathrm{mode}}(\phi)]=\begin{bmatrix} \cos^2\phi & 0 & \frac{\sin 2\phi}{\sqrt{2}} & 0 & \sin^2\phi & 0 \\ 0 & \cos^2\phi & 0 & \frac{\sin 2\phi}{\sqrt{2}} & 0 & \sin^2\phi \\ \frac{-\sin 2\phi}{\sqrt{2}} & 0 & \cos 2\phi & 0 & \frac{\sin 2\phi}{\sqrt{2}} & 0 \\ 0 & \frac{-\sin 2\phi}{\sqrt{2}} & 0 & \cos 2\phi & 0 & \frac{\sin 2\phi}{\sqrt{2}} \\ \sin^2\phi & 0 & \frac{-\sin 2\phi}{\sqrt{2}} & 0 & \cos^2\phi & 0 \\ 0 & \sin^2\phi & 0 & \frac{-\sin 2\phi}{\sqrt{2}} & 0 & \cos^2\phi \end{bmatrix}$$

Polarization rotation matrix for $N=1$

$$[\mathrm{rot}_{\mathrm{mode}}(\phi)]=\begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 & 0 \\ -\sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & \cos(\phi) & \sin(\phi) \\ 0 & 0 & -\sin(\phi) & \cos(\phi) \end{bmatrix}$$

Polarization rotation matrix for $N=2$

$$[\mathrm{rot}_{\mathrm{mode}}(\phi)]=\begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi & 0 & 0 \\ 0 & 0 & -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

tum of light beams is a rapidly expanding field and it is not yet clear where it may lead. It is possible that there will be other applications where the interaction of polarization and orbital angular momentum is best examined within a single formalism. It is therefore of considerable interest to extend the matrices describing the mode composition to include simultaneously the polarization behavior previously described by Jones matrices. Such matrices would then be able to model the propagation of both spin and orbital angular momentum, through an optical system comprising both polarizing and mode transforming components.

Any combination of polarized modes of order N can be expressed as a $2(N+1)$ element column vector containing the complex weightings of orthogonally polarized Hermite Gaussian modes of the same order, as with

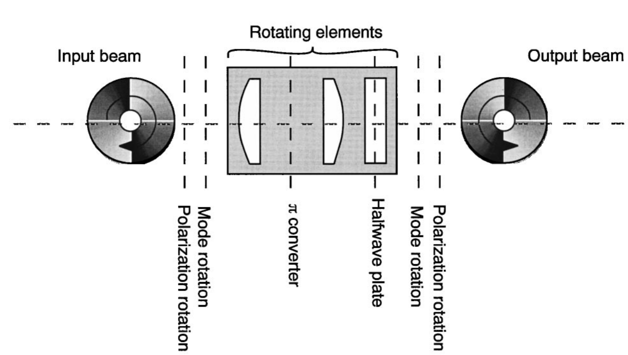

FIG. 2. Experimental configuration for the observation of the rotational frequency shift. (Arrows and shading on the mode distributions indicate the polarization and the sense of the azimuthal phase, respectively.)

$$\begin{bmatrix} a_{N,0}(\leftrightarrow) \\ a_{N,0}(\updownarrow) \\ a_{N-1,0}(\leftrightarrow) \\ a_{N-1,0}(\updownarrow) \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ a_{1,N-1}(\leftrightarrow) \\ a_{1,N-1}(\updownarrow) \\ a_{0,N}(\leftrightarrow) \\ a_{0,N}(\updownarrow) \end{bmatrix},$$

where the arrows in brackets denote the orientation of the polarization. The intensity is simply given by the square of the modulus of the vector representing the beam.

The $[2(N+1)\times 2(N+1)]$ matrices for the mode converters are based on those obtained previously, but each term must be repeated along the diagonal so that both polarizations are rephased. The $[2(N+1)\times 2(N+1)]$ matrices for waveplates are based on the corresponding Jones matrices, but the whole of each matrix must be repeated along the diagonal to act on each mode independently.

The rotation matrices are more complicated. The mode may be rotated with an image rotator and the polarization independently rotated by the use of a waveplate. The mode rotation matrix includes the same terms as our N dependent matrix given in Table I, but they must be diagonally repeated to rotate modes of both polarizations and interspaced with zeros to prevent their mixing. Similarly, the matrix for the rotation of the polarization is identical to that used within the Jones matrix formulation, but is diagonally repeated to rotate the polarization state of each of the individual Hermite Gaussian modes. The matrices for mode converters, waveplates, mode filters, polarizers and rotations are given in Table II.

To confirm the effectiveness of the matrix formulation, we modelled our rotational frequency shift experiment [10] for a circularly polarized, $\sigma=\pm 1$, Laguerre Gaussian mode. This enables us to find the frequency shift which results from the rotation of a beam containing both spin and orbital angular momentum. In the experiment, beam rotation was introduced by simultaneously rotating a π converter and half-wave plate; see Fig. 2. The matrix description of the output beam becomes

$$[\mathrm{rot}_{\mathrm{pot}}(\phi)]\times[\mathrm{rot}_{\mathrm{mode}}(\phi)]\times[W(\lambda/2)]\times[C(\pi)]$$
$$\times[\mathrm{rot}_{\mathrm{mode}}(-\phi)]\times[\mathrm{rot}_{\mathrm{pot}}(-\phi)]\times[LG_{l,p}(\sigma=\pm 1)].$$

A change in the handedness of the circularly polarized light, such that the spin angular momentum component is either additive or subtractive from the orbital angular momentum, gives a total angular momentum of the beam of $(l\pm 1)\hbar$ per photon. The frequency shift may be correctly deduced by comparison with its nonrotated equivalent to be $(l\pm 1)\Omega$, where Ω is the angular frequency of the beam. This offers powerful evidence of the utility of the matrices.

We have shown that there is a set of matrices equivalent to the Jones matrices which can express the behavior of light possessing orbital angular momentum and have developed them to account simultaneously for polarization. These matrices can be used in any optical system which conserves mode order. The equivalent of the Mueller 4×4 matrices, which depend upon the four Stokes parameters, for orbital angular momentum is currently the basis of further investigation.

It is a pleasure to acknowledge the contribution of Professor S. M. Barnett who drew to our attention to the rotation matrix in Ref. [12].

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